Quantum mechanics from a stochastic least action principle

Foundational Questions Institute Essay

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Abstract In this paper we begin by discussing the ultimate possibilities in physics. We then continue with a brief review of quantum mechanics and the history of action principles including a recent proposition of a universal action reservoir. We then move on to define a new least action principle for classical physics: a stochastic least action principle. We show that this principle yields the same result as Feynman's approach to quantum mechanics. We also show that it corresponds to classical physics in the case of $S >> \hbar$. As an example of the stochastic least action principle we apply it to the Einstein-Hilbert action of general relativity and obtain a form of quantum gravity. We also hint a stochastic differential equation approach to action based on the stochastic least action principle. In conclusion we discuss possible connections between our approach and other theories. Finally we return to discuss the open possibilities in physics.

1 Introduction

1.1 Possibilities of physics

What is ultimately possible in physics? If we had a theory of everything (TOE) that fulfills all necessary criteria, perhaps then could we have an answer to that question. Present day physics has theories that are mutually incompatible, in particular the standard model coupled with quantum mechanics is not compatible with general relativity. Thus we need a new theory that unifies these theories. But even if we could obtain a theory that only unifies parts of them we have made great progress. A possible road to the success in this pursuit might be to search for new principles of physics.

1.2 Quantum mechanics

Quantum mechanics with its many interpretations is the root cause of many open questions in physics. A nice definition of quantum mechanics can be said as follows (according to Lisi [7]):

> *Anything that can happen, does.*

Ultimately the rest of physics has to tell what can and what can not happen. But a future correct, perhaps deterministic, version of quantum systems perhaps could say what is ultimately possible in physics and what is not.

1.3 Minimal principles

The minimal principles of physics are many. They date back to the second century B.C. when Hero of Alexandria pointed out that when a light ray is reflected it takes the shortest path (this turned out not to be able to yield a correct law of refraction though [8]). Almost two millennia later Fourier made an attempt at a minimal principle when he postulated that a ray of light always travels from one point to another in a medium by a path that requires the least time. This law worked fine and indeed led to the law of reflection as well as Snell's law of refraction [8]. The first construction of a minimizing principle in mechanics was made by Mauperitius, who came along in 1747 and claimed that minimization of action was the key and that it was done with the wisdom of god [8]. Maupertius principle was later given a mathematical foundation by Lagrange. But it was Hamilton's principle, published in 1834 and 1835, that was to conquer them all and still stand as a corner stone of physics. This will be reviewed below.

2 Classical action

Classical action is perhaps best explained via Hamilton's principle. It can be stated as follows [8]:

> *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.*

The difference between the kinetic and potential energies is the lagrangian $L = T - U$. If one integrates this over time one gets the action $S = \int Ldt$. The actual path of the system is evaluated by Hamilton's principle (the variational principle) on the lagrangian, which in many cases turns out to be Lagrange's equation of motion:

$$\frac{dL}{dq} - \frac{d}{dt}\frac{dL}{d\dot{q}} = 0, \tag{1}$$

for some generalized coordinates q_i and their time derivatives $\dot{q}_i$.

3 Feynman's quantum mechanics

The perhaps best (or most useful) ansatz to quantum mechanics is the path integral formulation made by Feynman in 1948 [1]. In general it is a non-relativistic theory where a particle is travelling from one point to another and it takes all possible paths to that point, or so to speak their quantum mechanical histories are added to create the quantum partition function:

$$Z = \sum_{Paths} e^{\frac{i}{\hbar}S[Path]}. \tag{2}$$

It has been shown that this approach to quantum mechanics is equivalent to the canonical formulation of quantum mechanics [1], so that it in fact is the wave function:

$$\Psi = \sum_{Paths} e^{\frac{i}{\hbar}S[Path]}. \tag{3}$$

It has been used in various fields of physics, it is perhaps most notable for its success in quantum gravity theories [2, 5].

4 Wick rotation and Lisi's universal action reservoir

It has been shown that the complex wave function of quantum mechanics can be related to an ensemble of states [6]. In particular the path integral formulation of quantum mechanics (3) is related to an ensemble of action. This is proven via what is called Wick rotation. A Wick rotation is basically the change of time from a real quantity to an imaginary one via analytic continuation which amounts to $t \to it$. What the rotation does is to transform Euclidean space into Minkowski space because Minkowski space appears to have imaginary time in its metric construction. The Wick rotation also works the opposite way around and Euclidean space may be recovered from the Minkowski space with the same rotation. If one performs the Wick rotation on the Feynman path integral approach one gets a canonical ensemble of actions:

$$Z = \sum_{States} e^{-\frac{1}{\hbar}S[State]}. \tag{4}$$

This led Lisi to define a principle for quantum systems [6]:

> *A quantum system is one in contact with a universal action reservoir, providing a known expected path action.*

And the expected path action derived by Lisi is:

$$\langle S \rangle = \sum_{Paths} P(path)S(path), \tag{5}$$

which is obtained by maximizing the entropy of the system, for more detailed information on Lisi's approach see [6].

5 The stochastic action principle approach

Stochastic least action principles have been used in chaotic dynamical systems [11]. We shall here attempt to redefine Hamilton's principle of action in physics as a stochastic least action principle:

> *All the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints) creates a canonical ensemble with* $T = \hbar/k_B$ *of the time integral of the difference between the kinetic and potential energies, the actual path followed is with probability among all paths that which minimizes the time integral of the difference between the kinetic and potential energies.*

or in short:

> *The actions of all possible paths from one point to another constitute an ensemble of possible actions for a physical system for which the expected action is the classical one.*

Note that $\hbar = h/2\pi$ is the reduced Planck's constant and k_B is Boltzmann's constant. Any classical action is subject to this principle. Lets assume that we have a classical theory with a defined classical action S_C between two points a and b. Then the stochastic least action principle suggests that we setup all possible actions S_i from point a to point b as a partition function:

$$Z = \sum_i e^{-\frac{S_i}{\hbar}}, \tag{6}$$

where Planck's constant is analogous to the thermodynamic temperature of a canonical ensemble ($i\hbar \to k_B T$). As Lisi points out: *being constant reflects its universal nature - analogous to an isothermal canonical ensemble* [6]. Then for a particular action S' from point a to point b its probability is by the definition of a canonical ensemble:

$$P(S') = \frac{1}{Z} e^{-\frac{S'}{\hbar}}, \tag{7}$$

and the classical action of the system is the expected action:

$$S_C = \langle S \rangle = \sum_i P(S_i) S_i, \tag{8}$$

which is the expected path action spoken of in Lisi's principle of a universal action reservoir [6]. If we apply Wick rotation to the partition function (6) we obtain:

$$Z = \sum_i e^{i\frac{S_i}{\hbar}}, \tag{9}$$

which according to the path integral formulation of quantum mechanics [1] is equivalent to the wave function of quantum mechanics:

$$\Psi = \sum_i e^{i\frac{S_i}{\hbar}}. \tag{10}$$

Here it is shown that the stochastic least action principle yields the same result as Feynman's path integral formulation of quantum mechanics. This principle generalizes Hamilton's principle and renders the universal action reservoir a consequence thereof. The correspondence to classical mechanics in this ansatz occurs when the particular expected path action $S_i \to \infty$, for which the probability of such a state becomes 1. A similar way is to let $\hbar \to 0$, which yields the same result. In conclusion $S >> \hbar$ makes the system classical.

6 Application to gravity

A possible application of the stochastic least action principle is in the general theory of relativity. General relativity is governed by the Einstein-Hilbert action [3]:

$$S = \frac{c^4}{4\pi G}\int (R + L_M)\sqrt{-g}d^4x \tag{11}$$

where R is the Ricci scalar, $\sqrt{-g}d^4x$ is the invariant 4-volume element and L_M is the term for matter fields (sources). If we apply the stochastic least action principle it creates the following partition function:

$$Z = \sum_{\mathit{All\ Actions}} e^{-\frac{c^4}{2G\hbar}\int (R+L_M)\sqrt{-g}d^4x}, \tag{12}$$

since $\hbar = h/2\pi$. If we apply Wick rotation to (12) we get:

$$\Psi = Z = \sum_{\mathit{All\ Actions}} e^{i\frac{c^4}{2G\hbar}\int (R+L_M)\sqrt{-g}d^4x}, \tag{13}$$

which is the corresponding quantum mechanical wave function in this loose approach to quantum gravity. Although (13) might be divergent in its entirety some of its approximations might not be. It is perhaps possible to change summation from *all actions* to some form of *all geometries* or *all fields* or *all topologies* as to connect to some existing form of quantum gravity theory [5].

7 Stochastic differential equation approach

A more rigorous approach to dynamics from the stochastic least action principle would be to construct a stochastic differential equation of action. If one assumes that $\mathbf{L}(t)$ is the stochastic lagrangian with distribution set by the stochastic least action principle, then the stochastic least action principle becomes a stochastic differential equation [10]:

$$\frac{d\mathbf{L}(t)}{dq} - \frac{d}{dt}\frac{d\mathbf{L}(t)}{d\dot{q}} = 0, \tag{14}$$

which only for the expected value $\langle \mathbf{L}(t) \rangle$ is equivalent to the classical Lagrange equation (1):

$$\frac{d\langle \mathbf{L}(t) \rangle}{dq} - \frac{d}{dt}\frac{d\langle \mathbf{L}(t) \rangle}{d\dot{q}} = 0. \tag{15}$$

The development of this concept, in particular as regards general variational methods, is left open for further studies.

8 Conclusions and Summary

We have in this paper suggested that the ultimate possibilities in physics could more clearly be visualized with a better foundation for quantum mechanics. In spirit of this vision we constructed a new principle for quantum mechanics: A stochastic least action principle. We reviewed the history of minimal principles in physics and their connection to action. We also reviewed Feynman's path integral approach to quantum mechanics and discussed Lisi's approach to a universal action reservoir. In the process of this we defined the new stochastic least action principle. That approach was shown to yield the same result as Feynman's path integral approach to quantum mechanics. In addition to this we also showed the correspondence to classical physics for $S >> \hbar$. As an example of the new principle we applied it to the Einstein-Hilbert action and obtained a form of quantum gravity. In addition to this we hinted a stochastic differential equation approach to action based on the stochastic least action principle as a more fruitful ansatz. Other connections to stochastic gravity and stochastic electrodynamics might be interesting [4]. It is possible that this stochastic least action principle is predicated by some deterministic theory like the De Broglie-Bohm theory for example [9]. In general, perhaps there exists some more fundamental relationship between information theory, relativity and quantum mechanics that might bring forth the ultimate quantum theory. This in turn might lead to finally realizing what is ultimately possible in physics.

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