What is The principle Of inclusion and Exclusion?

Suppose a Survey is taken at an academic conference by 100 Scholars. 50 Can be grouped as physicists, 40 as mathematicians, 30 as philosophers. More precisely 25 are both physicists and mathematicians, 3 have physics, math and philosophy degrees, 5 are both mathematicians and philosophers, 10 are physicists and philosophers.

How many scholars do not have physics, math or philosophy degrees?

$P \equiv$ Physicist | $M \equiv$ Mathematician | $K \equiv$ Philosopher | $S \equiv$ Total number of scholars

$|S| = 100$

$|P| = 50$

$|M| = 40$

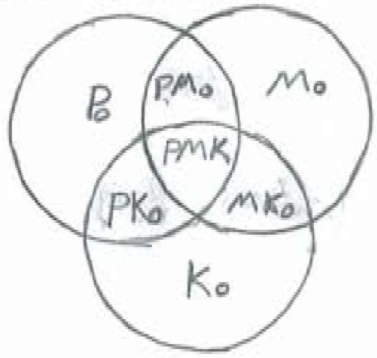
$|K| = 30$

$|PM| = 25$

$|PK| = 10$

$|MK| = 5$

$\boxed{|PMK| = 3}$



First we start by realizing that 3 scholars have triple degrees in physics, math, and philosophy, $PMK \equiv 3$. From this we can find the number of scholars that have only two degrees: Physics and Math (PM_o), Physics and Philosophy (PK_o) and math and philosophy (MK_o).

$|PM_o| = |PM| - |PMK|$

$|PM_o| = 25 - 3$

$\boxed{|PM_o| = 22}$

$|PK_o| = |PK| - |PMK|$

$PK_o = 10 - 3$

$\boxed{|PK_o| = 7}$

$|MK_o| = |MK| - |PMK|$

$|MK_o| = 5 - 3$

$\boxed{|MK_o| = 2}$

Now that we know the number of Scholars with exactly 3 degrees and exactly two degrees we can find the number of scholars with only a physics degree (P_o), math degree (M_o) or a philosophy degree (K_o)

$|P_o| = |P| - |PM_o| - |PK_o| - |PMK|$

$|P_o| = 50 - 22 - 7 - 3$

$\boxed{|P_o| = 18}$

$M_o = M - MK_o - PM_o - PMK$

$M_o = 40 - 2 - 22 - 3$

$\boxed{|M_o| = 13}$

$K_o = K - PK_o - MK_o - PMK$

$K_o = 30 - 7 - 2 - 3$

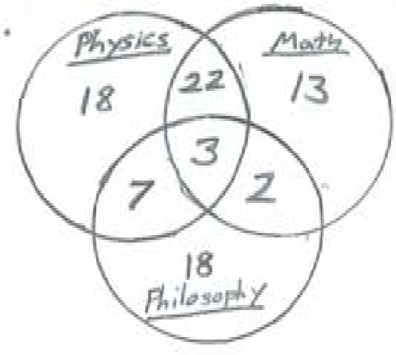
$\boxed{|K_o| = 18}$

Finally we can find the number of Scholars who do not have a physics, math or philosophy degree(s). I will call that number S_o.

$|S_o| = |S| - |P_o| - |M_o| - |K_o| - |PK_o| - |MK_o| - |PM_o| - |PMK|$

$|S_o| = 100 - 18 - 13 - 18 - 7 - 2 - 22 - 3$

$|S_o| = 17$



Out of 100 Scholars 17 do not have physics, math or Philosophy degrees; nor any Combination of those degrees.

This problem initially looks easy, but what makes it hard is taking into consideration the various combination of degrees a scholar can have. There is a much faster way to solve this problem.

First notice that adding the elements of the sets P, M + K produce a number larger than the number of elements in the set S

$|S| = 100$

$|P|+|M|+|K| = 50+40+30 = 120$

Why is $|P|+|M|+|K| > |S|$? It is because we count some sets twice such as $|PM_0|$, $|PK_0|$, $|MK_0|$ twice and we count the set $|PMK|$ three times:

$$|P|+|M|+|K| = \underbrace{|P_0|+|PM_0|+|PK_0|+|PMK|}_{|P|} + \underbrace{|M_0|+|MK_0|+|PM_0|+|PMK|}_{|M|} + \underbrace{|K_0|+|PK_0|+|MK_0|+|PMK|}_{|K|}$$

$$|P|+|M|+|K| = |P_0|+|M_0|+|K| + 2|PM_0|+2|PK_0|+2|MK_0|+3|PMK|$$

Ideally I would like to solve this problem quickly given only the sets; $|S|, |P|, |M|, |K|, |PM|, |PK|, |MK|$ and $|PMK|$ which were given at the beginning and required no additional calculations to get $|PM_0|$, etc. Since I counted some sets twice if I subtracted the sets once from $|P|+|M|+|K|$ then in actuality I counted them $2-1$ = one time.

$$|P|+|M|+|K| - |PM| - |PK| - |MK| = |P|+|M|+|K| - \underbrace{(|PM_0|+|PMK|)}_{|PM|} - \underbrace{(|PK_0|+|PMK|)}_{|PK|} - \underbrace{(|MK_0|+|PMK|)}_{|MK|}$$

As you can see $|P|+|M|+|K|-(|PM|+|PK|+|MK|)$ counts $|PM_0|$, $|PK_0|$, $|MK_0|$ once, but $|P|+|M|+|K|$ counts $|PMK|$ 3 times and $|PM|+|PK|+|MK|$ counts $|PMK|$ 3 times. $(|P|+|M|+|K|)-(|PM|+|PK|+|MK|)$ counts $|PMK|$ $(3)-(3)=0$ times. This means that Therefore we add the set $|PMK|$ and are able to get the correct number of scholars that have either a physics, math or philosophy degree:

$$|P \cup M \cup K| = (|P|+|M|+|K|) - (|PM|+|PK|+|MK|) + (|PMK|)$$

$$|P \cup M \cup K| = (50+40+30) - (25+10+5) + (3)$$

$$|P \cup M \cup K| = (120) - (40) + (3) = 83$$

If the conference has 100 scholars $(S=100)$ and 83 scholars have either a physics, math, or philosophy degree or some combination of those degrees $(|P \cup M \cup K| = 83)$ then the number of scholars without those degrees is:

$$|S_0| = |S| - |P \cup M \cup K| = 100 - 83 = 17 \checkmark$$

These types of problems are easy if we can group things in a small number of ways. In order to solve problems where the number of groups and combinations are large we must create a general method.

Essentially the Principle of Inclusion and Exclusion is counting...

The Principle of Inclusion and Exclusion

Suppose we have a collection of n distinct objects which have one or more properties we will label as, $P_1, P_2, \ldots, P_{z-1}, P_z$.

$N_0 \equiv$ The number of objects that have none of the properties.

$N(P_i) \equiv$ The number of objets that have the i^{th} property, P_i.

$N(P_iP_j) \equiv$ The number of objects that have both the i^{th} + j^{th} properties, P_i + P_j.

$N(P_{i_1} \ldots P_{i_m}) \equiv$ The number of objects that have a combination of the properties $(P_{i_1}, P_{i_2}, \ldots, P_{i_m})$

$$\boxed{N_0 = N - \sum N(P_i) + \sum N(P_iP_j) - \underbrace{\sum N(P_iP_jP_k)}_{\substack{\text{The number of} \\ \text{objects that} \\ \text{satisfy the } i^{th} \\ j^{th} + k^{th} \text{ property}}} + \cdots + \underbrace{(-1)^m \sum N(P_{i_1} \ldots P_{i_m})}_{\substack{\text{The number of objects} \\ \text{that satisfy any number} \\ \text{of } m \text{ properties that also} \\ \text{satisfy the } i^{th} \text{ property}}} + \cdots + \underbrace{(-1)^z N(P_1P_2 \cdots P_z)}_{\substack{\text{The term for the} \\ \text{number of objects} \\ \text{that satisfy all the} \\ \text{properties.}}}}$$

We finally have the equation for the derangement! (Oops I mean n_i)

$N = n! \equiv$ The total number of permutations $(n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1)$ for any set.

$N_0 = n_i \equiv$ The total number of permutations in which NO object remains in its original position.

$P_i \equiv$ The property that the i^{th} element of the set $(1\ 2 \cdots i \cdots n)$ remains in its original position.

$N(P_i) \equiv$ The number of permutations where the i^{th} element remains in its original position and $N(P_i) = (n-1)!$ because i is one element and $(n-1)!$ is the number of permutations where i is fixed. In other words out of n objects we choose that object i is fixed and arrange the remaining $(n-1)$ objects.

Similarly:

$N(P_iP_j) = (n-2)!$ and $N(P_iP_jP_k) = (n-3)!$

Therefore by applying this to the equation for the principle of inclusion and exclusion we get:

$$n_i = n! - \sum_i (n-1) + \sum_{ij} (n-2)! - \sum_{ijk} (n-3)! + \cdots + (-1)^m \sum_{i_1 \ldots i_m} (n-m)! + \cdots + \underbrace{(-1)^n}_{\text{Note } (n-n)! = 0! = 1}$$

What is $\sum_i (n-1)!$?

It means we add $(n-1)!$ n times $\left((n-1)!_1 + \cdots + (n-1)!_n\right) = n(n-1)!$

We can rewrite $n(n-1)!$ using the binomial coefficent as:

$$\binom{n}{1}(n-1)! \Rightarrow \left(\frac{n!}{1!(n-1)!}\right)(n-1)! \Rightarrow \left(\frac{n\cdot(n-1)!}{(n-1)!}\right)(n-1)! \Rightarrow n(n-1)!$$

Similarly:

$$\sum (n-2)! = \binom{n}{2}(n-2)!$$

$$\sum (n-3)! = \binom{n}{3}(n-3)!$$

$$\sum (n-m)! = \binom{n}{m}(n-m)!$$

Writing the $\sum (n-m)!$ in terms of the binomial coefficents $\binom{n}{m}(n-m)!$ allows us to rewrite:

$$n_i = n! - \sum_{i} (n-1)! + \sum_{ij} (n-2)! - \sum_{ijk} (n-3)! + \cdots + (-1)^m \sum_{i_1 \ldots i_m} (n-m)! + \cdots + (-1)^n$$

To an equation with binomial coefficients:

$$n_i = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^m \binom{n}{m}(n-m)! + \cdots + (-1)^n$$

We can simplify the equation by using summation notation as:

$$n_i = \sum_{m} (-1)^m \binom{n}{m} (n-m)!$$

Lets examine $\binom{n}{m}(n-m)!$:

$$\binom{n}{m}(n-m)! = \frac{n!}{m!\cancel{(n-m)!}} \cancel{(n-m)!} = \frac{n!}{m!}$$

therefore $n_i = \sum_{m} (-1)^m \binom{n}{m} (n-m)!$ becomes:

$$\boxed{n_i = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!}}$$

The Taylor expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ ($x = -1$ for the derangement)

$$n_i = n! e^{-1}$$

In general the number of permutations of an n element set in which no object remains in its original position is:

$$\boxed{n_i = \left[\frac{n!}{e}\right]}$$

($[\ ]$ is the nearest integer function)

The probability any permutation $(n!)$ is a derangement (n_i) is:

$$\frac{n_i}{n!} \doteq e^{-1} = 1/e = 0.368 \Rightarrow$$ 36.8% of the time any permutation is a derangement.

The probability that the first permutation is a derangement is: $Prob(1) = \frac{n_i}{n!}$ and the probability a second permutation is a derangement is $Prob(1,2) = Prob_1(2) = \frac{n_i}{n!}$.

Because $Prob(1) = Prob_1(2)$ we say that the events are independent of each other:

$$Prob(1,2) = \frac{n_i}{n!} \cdot \frac{n_i}{n!} = \left(\frac{n_i}{n!}\right)^2 = e^{-2}$$

In general then:

$$\boxed{Prob(1, 2, \ldots, n) = e^{-n}}$$

Any permutation of n objects like an einstein solid with q dots of energy and n bars of oscillators $(\bullet\bullet\bullet|\bullet\bullet|\bullet)$ can be understood using derangements and partial derangements (see my essay).

$-B^2$:)

$$\boxed{\text{Entropy} \equiv \text{Derangements}}$$