

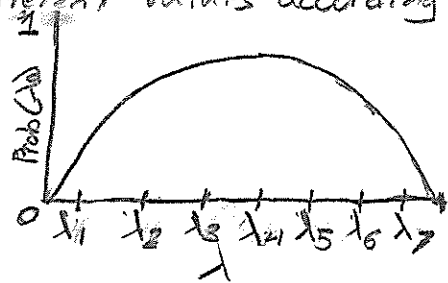
What is the collapse of the wavefunction?

An observable is initially in a superposition of different values according to

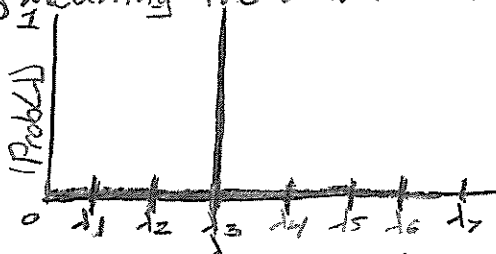
$$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \phi_n(x) e^{-i\omega_n t}$$

The probability the eigenvalue λ_n is found is;

$$|\Psi(x,t)|^2 = C_n^2$$



We perform a "Measurement" of the energy and find $E_3 = \hbar\omega_3$.
If we perform an additional measurement we find the same energy eigenvalue $E_3 = \hbar\omega_3$ with certainty, meaning the wavefunction has collapsed to that value.



If the second "Measurement" is not made "fast" enough then the wavefunction starts to spread out according to the time-dependent Schrödinger equation:

$$H\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

Using separation of variables we know the wavefunction evolves according to:

$$f(t) = e^{-i\omega t}$$

What is the Measurement Problem?

The linear equation $f(t) = e^{-i\omega t}$ implies that we know the wavefunction at any time if we know what it is at $t=0$.

There is an imaginary frequency ($\omega = 2\pi \nu$) measured in Hertz (ν) multiplied by a time measured in seconds (s). This results in an increasing imaginary number that denotes time or a decreasing imaginary number for T-symmetry.

Time in any physics equation is assumed to have an order such that an initial condition t_i comes before a final condition t_f . If time is imaginary it is not possible for it to be ordered. In particular for the complex numbers to be an ordered field then they must be able to satisfy the order axioms including:

If $a < b$ and $0 < c$ then $ac < bc$

$0 \equiv 0 + 0i$ and $i \equiv 0 + 1i$ which means $0 < i$ ($z = a + bi$ $a, b \in \mathbb{R}$)

$-i < i$ and $i = i$ then $| < - |$ false

To produce the ordered field we use the inner product $\langle \Psi | \Psi \rangle$ to map imaginary numbers to the real numbers where they can be ordered $\mathbb{C} \xrightarrow{|\cdot|} \mathbb{R}$.

What is the first Wave equation a physics student learns?

$$\Psi(x,t) = A \cos(kx - \omega t + \delta) \quad \left| \begin{array}{l} A \equiv \text{amplitude} \\ k \equiv \frac{2\pi}{\lambda} \quad \lambda \equiv \text{wavelength} \\ \omega \equiv 2\pi\nu \quad \nu \equiv \text{frequency} \\ \delta \equiv \text{Phase} \end{array} \right.$$

Then comes Euler's formula:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Using Euler has several advantages because e^{-x} is easy to differentiate $\frac{d}{dx} e^{-x} = -e^{-x}$ and integrate (integration by parts). e^{-x} is also easy to multiply together $e^{-x} \cdot e^{-x} = e^{-2x}$. Trig functions are not as easy or as fast to manipulate because they require us to use trig identities. Also $\frac{d}{dx} \cos(x) = -\sin(x)$ and $\frac{d}{dx} \sin(x) = \cos(x)$ a frequent and simple mistake is placing the "-" sign on the wrong trig derivative. Using Euler's formula cuts down on simple mistakes and does not require any trig identities. We must remember to take the real part of $e^{i\theta}$ to reclaim our wave equation:

$$\Psi(x,t) = \text{Re}(e^{i\theta}) = \text{Re}(\cos(\theta) + i \sin(\theta)) = \cos(\theta)$$

So to reproduce the familiar wave equation $\theta \equiv kx - \omega t$. What about the δ ? $e^{i\theta} \cdot e^{i\delta} = e^{i(\theta + \delta)}$; $e^{i\delta} \equiv$ global phase factor.

This now becomes the plane wave equation seen in ETM and optics:

$$\Psi(x,t) = A e^{i[k(x - vt) + \delta]} \quad (1)$$

To understand how the plane wave equation evolves (1) we use the differential wave equation:

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2} \quad (2)$$

Plugging (1) into (2) gives:

$$\frac{\partial^2}{\partial x^2} (A e^{i[k(x - vt) + \delta]}) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} (A e^{i[k(x - vt) + \delta]})$$

$$-A^2 k^2 \Psi(x,t) = \frac{1}{v^2} A^2 \omega^2 \Psi(x,t)$$

$$\boxed{v^2 k^2 = \omega^2} \quad (3)$$

$$\omega = vk \Rightarrow \lambda \nu = \text{Velocity}$$

Equation (2) is true for both $\cos(kx - \omega t)$ and $e^{i[k(x - vt) + \delta]}$ out of simplicity physicists used $e^{i[k(x - vt) + \delta]}$ form until it was an unbreakable habit. This works fine for ETM but what about quantum?

Before going into quantum we must ensure that classical physics is possible from quantum physics:

Classical Physics

Energy: $E = \frac{1}{2}mv^2 + C$
 Momentum: $P = MV$

V is the velocity and " $\frac{1}{2}mv^2$ " is called the kinetic energy. C is the potential energy and for now we will safely assume $C=0$. E is the mechanical energy and it is conserved.

How can we write energy in terms of momentum?

$$E = \frac{p^2}{2m} \quad \left| \quad \frac{p^2}{2m} = \frac{m^2v^2}{2m} = \frac{1}{2}mv^2 \checkmark \right.$$

So the classical energy written using the classical momentum is:

$$E = \frac{p^2}{2m} \quad (4)$$

What are the equations for energy and momentum in quantum mechanics?

Quantum Physics

Energy: $E = \hbar\omega$ (5) $\hbar \equiv \frac{h}{2\pi}$ and h is called Planck's constant it has units of Joule seconds (J.s). $\omega \propto K$??
 Momentum: $P = \hbar K$ (6) Wave-particle duality of quantum.

We know that eventually quantum must give the classical energy $E = \frac{p^2}{2m}$. Substituting the quantum equations (5) + (6) into the classical equation (4) gives:

$$\hbar\omega = \frac{\hbar^2 K^2}{2m} \quad (7) \quad (\omega \propto K^2)$$

So quantum uses wavefunctions $\psi(x,t) = A e^{i[\hbar Kx - \hbar\omega t]}$. Notice if we use the differential wave equation (7) for our quantum wavefunction we will end up with (3) where $\omega^2 \propto K^2$. In order to get the classical equation for energy (4) we must have a differential wave equation that gives $\omega \propto K^2$?

How do we construct a differential wave equation that has $\omega \propto K^2$?

Notice:

$$\frac{\partial \psi(x,t)}{\partial t} \Rightarrow \omega \quad \vee \quad \frac{\partial^2 \psi(x,t)}{\partial x^2} \Rightarrow \omega^2$$

This means our differential wave equation must be first order with respect to time, $\frac{\partial \psi(x,t)}{\partial t}$, and second order with respect to space, $\frac{\partial^2 \psi(x,t)}{\partial x^2}$.

$$C_1 \frac{\partial \psi(x,t)}{\partial t} = C_2 \frac{\partial^2 \psi(x,t)}{\partial x^2} \quad (8) \quad | \quad C_1, C_2 \text{ are constants}$$

Plugging (1) into (8) gives:

$$-C_1 A^2 \omega \psi(x,t) = -C_2 A^2 K^2 \psi(x,t) \Rightarrow C_1 \omega = C_2 K^2$$

Now set $C_1 = \hbar$ + $C_2 = -\frac{\hbar^2}{2m}$

Next Plug $C_1 = \hbar$ + $C_2 = -\frac{\hbar^2}{2m}$ back into equation (8):

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} \quad (9)$$

Equation 9 assumes our potential energy is zero (Free Particle Equation).

If we include the potential energy then equation 9 becomes the Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + C\Psi$$

Most textbooks rename C as V and include the y+z derivatives:

Schrödinger Equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi \quad (10)$$

($\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$) the left hand side is the energy, E, and the right hand side is called the Hamiltonian, H. This gives the short hand: $E\Psi = H\Psi$ for the Schrödinger Equation.

The Schrödinger equation is first order with respect to time. The common thinking is that this means the state of a system at time t_0 determines its future behavior. When the potential V is independent of time solutions to the Schrödinger equation can be found using Separation of Variables:

$$\Psi(x,t) = \phi(x) f(t)$$

Because the Schrödinger Equation is first order with respect to time this leads to

$$f(t) = e^{-i\omega t} \quad (11)$$

It is well known that solutions to linear homogeneous equations can be found using the eigenvalue equation from linear algebra:

$$\frac{dY}{dt} = AY = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y$$

$\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ $n \times 1$ Column vector
 $\begin{pmatrix} a & \dots & n \end{pmatrix}$ $1 \times n$ Row vector

then find λ for $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc = 0$ $1 \times 1 = \text{scalar!!}$

Eigenvalues have the "characteristic" property that: $A_n Y = \lambda Y$ still $n \times 1$ matrix

Once you solve for an eigenvalue and its corresponding eigenvector you can construct the straight line solution:

$$AY_n = \lambda_n Y_n \Rightarrow Y(t) = e^{A_n t} Y_n$$

In quantum we have $H\Psi_n = E_n\Psi_n$ H is called an observable it gives the energy Eigenvalue E_n that corresponds with the eigenvector Ψ_n . So if we want to know every possible value Ψ can have then we write it as a superposition of every possible eigenvalue/eigenvector.

$$\Psi_n(x,t) = \phi_n(x) e^{-i\omega_n t}$$

$$\Psi(x,t) = \sum c_n \phi_n(x) e^{-i\omega_n t} \quad (12)$$

So how does equation (12) give us the probabilities?

To get the probabilities you have to square it or in general take an inner product. (This causes problems)

The Problem

$$P_{bb} = \Psi^\dagger \Psi = \int \Psi^* \Psi = \int C_n \cdot C_n \phi_n(x) \cdot \phi_n(x) e^{-i\omega_n t} \cdot e^{i\omega_n t} = C_n^2 \phi_n^2 = C_n^2 \phi_n^2 = C_n^2 \phi_n^2$$

Notice that somehow time annihilated itself...

What does it all mean?

It means there is a chance I will observe the system in a state with some measurable value and time doesn't come into play because it "drops" out.

- Measurable Observable = $n \times n$ Hermitian Matrix (If $A^\dagger = A$ then A is Hermitian)
- Chance = C_n^2 (the probability)
- State = ϕ_n eigenvector / eigenstate
- Value = λ_n eigenvalue

Recall

$$A \phi_n = \lambda_n \phi_n$$

$$H \phi_n = E_n \phi_n \text{ (The time-independent Schrodinger equation)}$$

The probability the system is "observed" to have the value E_n (state ϕ_n) is C_n^2 .

What happened to the ϕ_n^2 ?

Remember ϕ_n is an eigenvector and a basis in a linearly independent vector space. Linearly independent bases (sp^3) are orthogonal which means that if a basis ϕ_1 and ϕ_2 are "multiplied" then a zero pops out and that probability coefficient disappears ($C_1 \phi_1 \cdot C_2 \phi_2 = C_1 C_2 \phi_1 \phi_2 = C_1 C_2 (0) = 0$). What if ϕ_1 is "multiplied" by ϕ_1 ? then the number 1 pops out ($C_1 \phi_1 \cdot C_1 \phi_1 = C_1^2 \phi_1^2 = C_1^2 (1) = C_1^2$)

This " $\phi_i \phi_j = 0 \neq \phi_i \phi_i = 1$ " idea is summarized as:

$$\langle \phi_i | \phi_j \rangle = \delta_{ij} \quad \left\{ \begin{array}{l} \text{if } i=j \text{ then } \delta_{ij} \Rightarrow 1 \\ \text{if } i \neq j \text{ then } \delta_{ij} \Rightarrow 0 \end{array} \right. \quad (13)$$

The δ_{ij} idea is standard physics and is used all the time in classical physics. For instance, take two vectors: $\vec{A} = a\hat{x} + b\hat{y} + c\hat{z}$ + $\vec{B} = d\hat{x} + e\hat{y} + f\hat{z}$ and take their "dot" product a.k.a. scalar product a.k.a. the inner product in a Euclidean space:

$$\vec{A} \cdot \vec{B} = (a\hat{x}\hat{x} + a\hat{y}\hat{y} + a\hat{z}\hat{z} + b\hat{x}\hat{y} + b\hat{y}\hat{y} + b\hat{y}\hat{z} + c\hat{x}\hat{z} + c\hat{y}\hat{z} + c\hat{z}\hat{z})$$

$$\vec{A} \cdot \vec{B} = (ad(1) + ae(0) + af(0) + bd(0) + be(1) + bf(0) + cd(0) + ce(0) + fc(1))$$

$$\vec{A} \cdot \vec{B} = (ad + be + fc)$$

What is the difference between orthogonal and orthonormal?

They are the same idea " $\langle \phi_i | \phi_j \rangle = \delta_{ij}$ " an orthonormal basis vector is a normalized orthogonal basis vector.

What is normalization?

Our wavefunction has an additional coefficient, "A" that we must solve for

$$\Psi(x,t) = A \sum c_n \phi_n e^{-i\omega_n t}$$

First note that the sum of all probabilities equals 1 because something must happen. $\sum \frac{1}{2}T + \frac{1}{2}H = 1$ (I flipped a coin it will eventually land on one side)

$$\sum c_n^2 = 1$$

Therefore $1 = A^2 \sum c_n^2 \Rightarrow A = \frac{1}{\sqrt{\int_{-\infty}^{\infty} \Psi^* \Psi}}$ (I made the discrete \sum the continuous \int which I should do)

What is the difference between $|\psi\rangle$ and $\langle\psi|$?

This is called Dirac notation $|\psi\rangle$ is called a ket and $\langle\psi|$ is called a bra. Kets are column vectors and bras are the complex conjugate transpose symbolized by † (dagger).

$$|\psi_n\rangle = \begin{pmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{jn} \end{pmatrix} \quad \dagger |\psi_n\rangle = \langle\psi_n| = (-i c_{1n}, -2i c_{2n}, \dots, -n i c_{jn})$$

This means $\langle\psi_n|\psi_n\rangle =$ a real number

Why are Hermitian Matrices used?

Because the eigenvalues of a Hermitian matrix are real:

$$A^\dagger = A \Rightarrow A = \begin{pmatrix} a_{11} & i b_{12} \\ -i b_{21} & d_{22} \end{pmatrix}$$

† = * † T Take the complex conjugate A^*

$$A^* = \begin{pmatrix} a_{11}^* & (i b_{12})^* \\ (-i b_{21})^* & d_{22}^* \end{pmatrix} = \begin{pmatrix} a_{11} & -i b_{12} \\ i b_{21} & d_{22} \end{pmatrix}$$

The complex conjugate of any real number λ is itself. $\lambda \in \mathbb{R}$ then $\lambda^* = \lambda$

Next take the transpose

$$\begin{pmatrix} a_{11}^T & -i b_{12}^T \\ i b_{21}^T & d_{22}^T \end{pmatrix} = \begin{pmatrix} a_{11} & i b_{21} \\ -i b_{12} & d_{22} \end{pmatrix} = \begin{pmatrix} a & i b \\ -i b & d \end{pmatrix} = A$$

∴ $A^\dagger = A$ and so A is Hermitian

This means if A is my Hermitian observable and contains my eigenvalues and $\langle\psi|\psi\rangle$ contains my probabilities then $\langle\psi|A|\psi\rangle$ contains everything.

In the end how do we go about finding the probability coefficients?

We use boundary conditions like in the infinite and finite square wells. For the free particle it is periodic boundary conditions / Fourier analysis (wavepacket)

The most striking thing about quantum is that it is essentially probability theory but combinatorics is never mentioned and complex numbers are everywhere. In probability theory combinatorics is always used and I have never seen probability theory use complex analysis instead of combinatorics.

Just for fun:

Cross out all the imaginary numbers in the wavefunction and the Schrödinger equations

$$\psi(x,t) = e^{i(kx - \omega t)} \Rightarrow e^{(kx - \omega t)}$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} \Rightarrow \hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

$$\hbar \frac{\partial e^{(kx - \omega t)}}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 e^{(kx - \omega t)}}{\partial x^2}$$

$$-\hbar \omega \psi(x,t) = -\frac{\hbar^2 k^2}{2m} \psi(x,t)$$

$$\hbar \omega = \frac{\hbar^2 k^2}{2m} \quad \left| \begin{array}{l} E = \hbar \omega \\ p = \hbar k \end{array} \right.$$

$$E = \frac{p^2}{2m} \quad \text{! !}$$

$$\psi(x,t) = \phi(x) e^{-i\omega t} \Rightarrow \psi(x,t) = \phi(x) e^{-\omega t} \quad \text{So } V?$$