

Notes on the quantum decay of Λ and baby universes

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We consider the decay of the de Sitter vacuum by quantum means and the prospect for this as a mechanism for the production of nascent cosmologies or baby universes. The observable universe, under eternal inflation from dark energy, will asymptotically evolve to a de Sitter spacetime. This spacetime is a vacuum configuration with a cosmological constant Λ . The stationary metric for this spacetime is

$$ds^2 = Adt^2 - A(r)^{-1}dr^2 - r^2d\Omega^2, \quad A(r) = (1 - \Lambda r^2/3)$$

A radial null geodesic with $ds^2 = 0$ and $d\Omega^2 = 0$ gives the velocity $\dot{r} = dr/dt = A(r)$, where this pertains to both out and in going geodesics near the cosmological horizon $r = \sqrt{3/\Lambda}$ as measured from $r = 0$. The total action for the motion of a particle is $S = \int p_r dr - \int H dt$. Consider the bare action of massless particles, using methods found in [1], across the horizon from r to r' ,

$$S = \int_r^{r'} p_r dr = \int_r^{r'} \int_0^{p_r} dp_r dr.$$

The radial velocity of a particle is $\dot{r} = dr/dt = dH/dp_r$, which enters into the action as,

$$S = \int_r^{r'} \int_0^H \frac{dH'}{\dot{r}} dr.$$

The field defines $H' = \hbar\omega'$. The integration over frequencies is from E to $E - \omega$, for the ADM energy. The action is properly written as

$$S = -\hbar \int_r^{r'} \int_E^{E-\omega} \frac{d\omega'}{\dot{r}} dr,$$

where the negative sign indicates the quanta is tunneling across the horizon to escape the Hubble region with radius $\sqrt{3/\Lambda}$. The radial velocity

$$\dot{r} = \sqrt{\Lambda/3}r$$

defines the action

$$S = -\hbar \int_r^{r'} \int_0^\omega \frac{d\omega dr}{\pm 1 - \sqrt{\Lambda r^2/3}} = \sqrt{3/\Lambda} \tanh^{-1}(\sqrt{\Lambda/3}r)$$

The action is then the delay coordinate evaluated as

$$r^* = \int \frac{dr}{1 - \Lambda r^2/3} = \sqrt{3/\Lambda} \tanh^{-1}(\sqrt{\Lambda/3}r).$$

The domain $[0, \sqrt{3/\Lambda})$ defines a real valued action. Since, $\tanh^{-1}(x) = \frac{1}{2} \ln((1+x)/(1-x))$ for $r > \sqrt{3/\Lambda}$ the argument of the logarithm is negative. In this case the action is

$$S = \sqrt{3/\Lambda} \ln\left(\frac{\sqrt{\Lambda/3}r + 1}{\sqrt{\Lambda/3}r - 1}\right) + i\pi\sqrt{3/\Lambda}.$$

The imaginary part represents the action for the quantum field emission as $r \rightarrow \infty$. The delay coordinate is defined on $[0, \infty)$ which assures an S-matrix is defined on an unbounded causal domain, and this holds in general as well.

The metric elements we need to be concerned about is then $g_{tt} = (1 - \Lambda r^2/3)$, $g_{rr} = 1/(1 - \Lambda r^2/3)$. The extrinsic tensor (Gauss' second fundamental form) is then $K_{ab} = \frac{1}{2}n^c \partial_c(g_{ab})$, for $n^r = \sqrt{A(r)}$. These are rather easy to evaluate this

$$K_{tt} = n^r \partial_r g_{tt} = \frac{2\Lambda r}{3} (1 - \Lambda r^2/3)^{1/2}$$

$$K_{rr} = n^r \partial_r g_{rr} = -\frac{2}{3} \frac{\Lambda r}{3} (1 - \Lambda r^2/3)^{-3/2}.$$

The curvature $G^{00} = (TrK)^2 - Tr(K^2)$ which is then

$$G^{00} = (K_{tt} + K_{rr})^2 - K_{tt}^2 - K_{rr}^2 = 2K_{tt}K_{rr} = \frac{8}{9} \left(\frac{\Lambda^2 r^2}{1 - \Lambda r^2/3} \right).$$

This is the vacuum energy available in the de Sitter vacuum.

We now perform a similar analysis above, but instead consider the radiation production according to the transition $\Lambda \rightarrow \Lambda + \delta\Lambda$. The transition is considered according to the metric back reaction of the de Sitter vacuum. This tunneling defines the imaginary part of the action

$$S = \int_{\Lambda_0}^{\Lambda} p dr.$$

The velocity term here is the computed from the Hamilton equation $\dot{r} = \partial H / \partial p$ which permits this to be written as

$$S = \int_{\Lambda_0}^{\Lambda} \int_0^R \frac{dr}{\dot{r}} dH.$$

The Hamiltonian used is the $H = G^{00}$ computed here, with $\dot{r} = \pm(1 - \Lambda r^2/3)$ with

$$\begin{aligned} S &= \int_{\Lambda_0}^{\Lambda} \int_{r_i}^{r_f} dr \left(\frac{4r}{3} \right)^2 \int_{\Lambda_0}^{\Lambda} \frac{\Lambda}{(1 - \Lambda r^2/3)^2} \left(1 + \frac{\Lambda r^2/3}{1 - \Lambda r^2/3} \right) d\Lambda \\ &= 16 \int_{r_i}^{r_f} dr \left(\frac{1}{r^2} (\log(\Lambda r^2 - 3) - \frac{3}{\Lambda r^2 - 3}) \right) \Big|_{\Lambda_0}^{\Lambda} \\ &= 16 (\ln(\Lambda r^2 - 3)/r - \sqrt{\Lambda/3} \operatorname{arctanh}(\sqrt{\Lambda/3}r) - 1/r) \Big|_{\Lambda_0}^{\Lambda} \Big|_{r_i}^{r_f}. \end{aligned}$$

This complicated expression is evaluated for $\Lambda = \Lambda_0 - \delta\Lambda$ with

$$S = 8 \left(\sqrt{\Lambda/3} \operatorname{arctanh}(\Lambda/3r) + \frac{r}{1 - 3/(\Lambda r^2)} + \frac{2}{\Lambda - 3/r^2} \right) \Big|_{r_i}^{r_f} \Delta\Lambda$$

Of course this solution exhibits a singularity for the radius extended across $\sqrt{3/\Lambda}$. We then consider the integration with respect to the small change in the cosmological constant

$$\begin{aligned} S &= 16 \int_{r_i}^{r_f} dr \left(\frac{1}{r^2} (\log(\Lambda r^2 - 3) - \frac{3}{\Lambda r^2 - 3}) \right) \Big|_{\Lambda_0}^{\Lambda} \\ &\simeq 16 \left(\frac{1}{\Lambda r^2 - 3} - \frac{3}{(\Lambda r^2 - 3)^2} \right) \delta\Lambda \end{aligned}$$

The calculus of residues and Cauchy integral formula gives this result in a very simple form for $z^2 = \Lambda r^2$ and we then left with the result

$$S = \frac{\sqrt{3}}{4} \pi \delta\Lambda$$

Now that we have that set up we propose a quantum model. The $G^{00} = H$ which under the Hamiltonian constraint is zero. Yet with quantum mechanics we propose an evolution equation $i\partial\Psi[g, \phi]/\partial t = G^{00}\Psi[g, \phi]$, where the deviation from zero in the Wheeler DeWitt equation $NH\Psi[g] = 0$ is due to the occurrence of a scalar field in the spacetime. For now on we will substitute H for G^{00} . Now we want to compute a transition

rate, which involves both a change in g and the internal scalar field ϕ . So we take a second derivative of $\Psi = \Psi[g, \phi]$ to get

$$-\frac{\partial^2 \Psi}{\partial t^2} = H^2 \Psi + i \frac{\partial H}{\partial t} \Psi.$$

We now express this according to $|\Psi\rangle$ and $|\Psi'\rangle$ by

$$H^2 \Psi = H \int d\mu(\Psi') |\Psi'\rangle \langle \Psi' | H | \Psi \rangle,$$

which is just a completeness sum $1 = \int d\mu(\Psi') |\Psi'\rangle \langle \Psi'|$ inserted between the two Hamiltonians. We then have a summation over transition probabilities as

$$\int d\mu(\Psi') |\langle \Psi' | H | \Psi \rangle|^2 = -\langle \Psi | \frac{\partial^2}{\partial t^2} | \Psi \rangle - i \langle \Psi | \frac{\partial H}{\partial t} | \Psi \rangle.$$

The first term on the right hand side depends only on the transition of the scalar field, which vanishes if the scalar field is initially zero. The last term will then depend upon $\partial\Lambda/\partial t$, and may be computed from the G^{00} above. These two terms represent the long wavelength Hawking-Gibbon radiation produced by the decay of the cosmological constant and the change in Hamiltonian due to the production of this radiation.